# Constructing New Numbers (ver.1)

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## Preface

This is a **replacement text** extracted from Chapter 1 of an in‑progress paper, *A New Foundation of Mathematics*. Chapter 2 will implement a λ‑calculus interpreter; Chapter 3 will give **formal definitions, axioms, and proofs** on a custom language executable on that interpreter. In this chapter we present the construction of new numbers while explicitly stating **symbols, types, and equivalence relations**, anticipating the later formalization.

Guiding principles

* Treat **generation procedures (directed processes)** as first‑class objects rather than values themselves.
* Name the “neighbor” that **does not reach the limit value** as an **equivalence class of procedures** via qlim.
* Keep the whole system **countable** while connecting naturally to arithmetic through a layered structure (the **continuous ordinal numbers** (M)), then define the **new reals** ().

## Notation and Conventions

* **Function application** is written f() (no spaces). Repetition uses f^k().
* **Multiplication** uses · (or × when convenient).
* **Line breaks inside formulas** are allowed for layout; terminate a displayed formula with a semicolon ;.
* **Pronunciations**: a## means “one step to the **next** (right) of a”, and a♭♭ means “one step to the **previous** (left) of a” (made precise below).
* The ellipsis … is **human‑readable only**; mathematical definitions use explicit iteration or limits.

## 1. Constructing the Natural Numbers

**Def. 1.1 (Naturals)** Represent a natural number by **how many times a function is applied**:

* (0 := f())
* (1 := f(f()))
* (2 := f(f(f())))
* … (in general, (n) is the number of repetitions of (f))

Remark. This is isomorphic to **Church numerals**. We use Arabic digits (n) for these naturals.

## 2. Constructing the Integers

**Def. 2.1 (Integers)** An integer is an ordered pair **(sign, natural)**: [ z := (, n),{0, 1}, n. ]

* Examples:
  + (+2 := (0,2) = (f(), f^3()))
  + (+1 := (0,1) = (f(), f^2()))
  + (0 := (0,0) = (f(), f()))
  + (-1 := (1,1) = (f^2(), f^2()))
  + (-2 := (1,2) = (f^2(), f^3()))

**Normalization.** Identify ((1,0)) with ((0,0)) (so (-0=0)).

## 3. Constructing the Rationals

**Def. 3.1 (Rationals)** A rational is an ordered pair **(integer, integer)** with a nonzero denominator: [ q := (p, r),r. ]

**Normal form.** Use reduced fraction with **positive denominator** as the class representative.

* Examples: (2/1 := (2,1)), (1/1 := (1,1)), (0/1 := (0,1)), (-1/1 := (-1,1)), etc.

## 4. Rational Near‑Limits (Naming the Neighbors)

Fix (a). We define the **“neighbor”** of (a) as an **equivalence class of directed procedures** that **do not reach** the classical limit value.

**Def. 4.1 (Upper representative sequences).** For (b\_{>0}), [ q\_n^{+}(a;b):=(n=1,2,3,) ] is **monotonically decreasing** to (a) with (a) as a **lower bound**. Dually, set (q\_n^{-}(a;b):=) for the lower side ((b>0)).

**Def. 4.2 (qlim and neighbor classes).** Take the **tail‑equivalence class** (defined below) of (q\_n^{+}(a;b)) or (q\_n^{-}(a;b)), and denote by (): [ a := *{n} q\_n^{+}(a;b), a :=* {n} q\_n^{-}(a;b). ] This class does **not** depend on the choice of (b>0).

**Notation 4.3 (Infinitesimal step operators).** Write one step to the upper neighbor by ## and one step to the lower neighbor by ♭♭: [ a,;:=; q\_n^{+}(a;b),a,;:=; q\_n^{-}(a;b). ] The lexicographic order will be [ < a^{(-2)} < a^{(-1)} < a^{(0)}{=}a < a^{(1)} < a^{(2)} < , ] where (a^{(k)}) will be represented via an isomorphism in §5 ((k)).

Caution. qlim **does not return a classical real value**; it names an **equivalence class of directed generation procedures**.

### 4.4 Lemma (Independence of (b))

**Claim.** For fixed (a) and any (b\_1,b\_2>0), the sequences (q\_n^{+}(a;b\_1)) and (q\_n^{+}(a;b\_2)) are **tail‑equivalent**, hence define the same qlim class. Likewise for the lower side. **Proof (formal).** Define tail‑equivalence () by: for sequences ((x\_n),(y\_n)), we have (xy) iff (>0,N,nN,|x\_n-y\_n|<). Put (x\_n=a+b\_1/n), (y\_n=a+b\_2/n). Then (|x\_n-y\_n|=|b\_1-b\_2|/n). For any (>0) choose (N>|b\_1-b\_2|/). Then (nN) implies (|x\_n-y\_n|<). Both sequences are monotone with lower bound (a). Hence the same qlim class. ∎

### 4.5 Lemma (Reindexing invariance)

**Claim.** For (k,c), (r\_n:=q\_{kn+c}^{+}(a;b)) is tail‑equivalent to (q\_n^{+}(a;b)). Likewise for the lower side. **Proof (formal).** With (s\_n=a+b/n) and (r\_n=a+b/(kn+c)), the difference (d\_n=r\_n-s\_n=-b((k-1)n+c)/(n(kn+c))) tends to 0. Hence (rs). Monotonicity holds since the denominator increases in (n). ∎

### 4.6 Definition (Neighbor classes)

Define the **upper** class (N^{+}(a)) as ({ q\_n^{+}(a;b)b>0}) modulo 4.4; denote it by (a##). Define the **lower** class (N^{-}(a)) analogously; denote it by (a,!).

Note. For this chapter we restrict representatives to ((nab)/n). Allowing wider monotone families would require extending the equivalence, which is unnecessary here.

### 4.7 Proposition (Tail‑equivalence is an equivalence relation)

**Claim.** () is an equivalence relation on the set of rational sequences. **Proof.** Reflexive: (|x\_n-x\_n|=0). Symmetric: (|y\_n-x\_n|=|x\_n-y\_n|). Transitive: use (/2) and (N=(N\_1,N\_2)). ∎

### 4.8 Lemma (Non‑emptiness and uniqueness of (N{+}(a),N{-}(a)))

**Claim.** For every (a), both (N^{+}(a)) and (N^{-}(a)) are non‑empty and uniquely determined as equivalence classes. **Proof.** Non‑empty: take (b=1). Uniqueness: 4.4 shows independence of (b); 4.5 shows reindexing invariance. ∎

### 4.9 Proposition (Well‑definedness of (aa##) and (aa,!))

**Claim.** For each (a), the elements (a##) and (a,!) are independent of representatives and define unique elements (a^{(+1)}, a^{(-1)}M). Thus the maps (S^{+}:M), (S^{-}:M) given by (S{+}(a)=a{(+1)}) and (S{-}(a)=a{(-1)}) are well‑defined. ∎

### 4.10 Proposition (Monotonicity and injectivity of (S{+},S{-}))

**Claim.** If (a<b) then (S{+}(a)<S{+}(b)) and (S{-}(a)<S{-}(b)); both maps are injective. **Proof.** In lexicographic order on (M), ((a,1)<(b,1)) and ((a,-1)<(b,-1)) for (a<b). Injectivity is immediate. ∎

### 4.11 Proposition (One‑sided boundedness and boundary at (a^{(0)}))

Let (C^+\_a:={a^{(+k)}k}), (C^-\_a:={a^{(-k)}k}). Then (a^{(0)}) is the **infimum** of (C^+\_a) and the **supremum** of (C^-\_a). In particular (C^+\_a) is bounded below by (a^{(0)}), and (C^-\_a) is bounded above by (a^{(0)}). ∎

## 5. Continuous Ordinal Numbers ((M))

**Def. 5.1.** [ M := {(,a)a, }, a^{()}. ] Intuition: each rational point (a) carries an **integer‑graded neighboring layer**.

**Order (lexicographic).** [ (a{(k)}<b{()})(a<b)(a=bk<). ]

**Addition and integer scaling.** [ a{(k)}+b{()}=(a+b){(k+)},ma{(k)}=(ma)^{(mk)} (m). ]

Intuitive model. (M,) with a **formal first‑order infinitesimal** (>0). Reading (a^{(k)}) as (a+k), the symbols ##/♭♭ correspond to adding/subtracting ().

**Examples (“adding/removing symbols”).** - (1{(-1)}+1{(+1)}=2^{(0)}) - (0{(+2)}-0{(+3)}=0^{(-1)}) - (2{(+2)}(1/2)=1{(+1)})  
(Note: division is **not** closed in (M); interpret this as scalar multiplication by a rational.)

**Non‑reducible examples (leaving the integer layer).** (1^{(+1)}/2), (1^{(+1)}/3), (1/1^{(+1)}), (1{(-1)}/1{(+1)}), etc.

Summary. (M) is an abelian group under addition with integer scaling, but **not** closed under general division. We remedy this by forming fractions (next section).

### 5.4 Coefficient rules (##/♭♭) and indivisibility

Define (S{+}(a):=a{(+1)}) ((=a##)), (S{-}(a):=a{(-1)}) ((=a,!)). Repetition: ((S{+}){k}(a)=a^{(+k)}), ((S{-}){k}(a)=a^{(-k)}) ((k)).

**Laws (linearity w.r.t. addition).** 1. (a{(k)}+b{()}=(a+b)^{(k+)}). 2. (ma{(k)}=(ma){(mk)}) ((m)).

**Indivisibility (integer gradedness).** **Prop. 5.4.1.** There is **no half‑step in (M)**: there is no (xM) with (2x=a^{(+1)}).  
*Proof (()-model).* Writing (x=r+m) with (r, m), we would need (2r=0) and (2m=1), impossible. ∎

**Cor. 5.4.2 (fractional steps live in CR).** In () an element like (a{(+1)}/(2{(0)})) exists (a “half‑step”), but it is **not** an element of (M).

### 5.5 Proposition (Compatibility with addition; ()-action)

1. For (a,b) and (i,j), (a{(i)}+b{(j)}=(a+b)^{(i+j)}).
2. Extend (S{+},S{-}) by (S{+}(a{(k)})=a^{(k+1)}), (S{-}(a{(k)})=a^{(k-1)}).

**Cor. 5.5.1 (linearity).** (S{+}(x+y)=S{+}(x)+y=x+S^{+}(y)) and analogously for (S^{-}).

**Cor. 5.5.2 (uniform shift invariance).** For any (), (a{(i+)}+b{(j-)}=a{(i)}+b{(j)}).

**Cor. 5.5.3 (standard form of differences).** (a{(i)}-a{(j)}=0^{(i-j)}).

### 5.6 Proposition (() acts on (M))

Define (:MM) by ((n,a{(k)})=a{(k+n)}). Then () is a group action; (S^{+}=(1,)), (S^{-}=(-1,)).

## 6. New Reals (())

**Def. 6.1.** [ :={;|;x,yM, y}/. ]

**Equivalence/normalization.** (1) (x/ymx/my) for (m{0}); (2) put the denominator in a fixed normal form (positive rational part, integer layer); (3) reduce the fraction.

**Operations (closed under (+,-,,)).** [ =, =, ()^{-1}= (x). ]

**Order.** Via the model (M) with (>0), extend the **lexicographic** order to ().

**Embedding and countability.** The map () is natural; () is countable (e.g., encoded by ()).

* Examples: (1=1{(0)}/1{(0)}), (2=2{(0)}/1{(0)}), (0=0{(0)}/1{(0)}).

Note. We **do not claim equality of values** with classical irrationals; “irrational‑like” here means “not rational” **in this structure**.

### 6.2 Lemma (()‑normal form)

For a CR element (x/y) with denominator in normal form and positive rational part, there exist unique ((A,B)) such that [x/y = A + B,.] Concretely, with (x=a^{(k)}), (y=c^{(n)}) ((c>0)) we have [A=a/c,B=(kc-an)/c^2.] *Proof.* Multiply by ((c-n)/c) and use (^2=0). ∎

### 6.3 Definition (Order via NF)

Let (:) send (x/y) to ((A,B)) as in 6.2. Define ((A,B)<(C,D)) iff ([A<C]) or ([A=CB<D]) (lexicographic). This yields a well‑defined total order on CR.

### 6.4 Proposition (Order‑preserving embeddings)

1. (\_M:M), (\_M(x)=x/1^{(0)}), is injective and order‑preserving; ((\_M(a^{(k)}))=(a,k)).
2. (*:), (*(a)=a{(0)}/1{(0)}), is injective and order‑preserving; ((\_(a))=(a,0)).

### 6.5 Proposition (Operations via NF)

If ((x)=(A,B)) and ((y)=(C,D)), then  
(i) ((xy)=(AC,,BD));  
(ii) ((xy)=(AC,,AD+BC));  
(iii) if (A) then ((x{-1})=(1/A,,-B/A2)).

### 6.6 Proposition (Monotonicity in CR)

1. If (x<y) then (x+z<y+z) for all (z).
2. If ((z)=(C,D)) with (C>0), then (x<yxz<yz). In particular, if (y>0) (positive constant term in NF) then (x/y < x’/yx<x’).
3. For (0<x<y) with positive constant terms, (1/y<1/x) (order reversal for reciprocals of positives).  
   *Proof.* Compare constant terms via NF; lexicographic order handles ties with the ()‑coefficients. ∎

Caution 6.6.1. Multiplying by a **negative** element reverses the order.

## 7. New Complex Numbers (())

**Def. 7.1.** For (a,b), [ a+bi := ((a,-b),(b,a)). ] **Operations** (the matrix representation recovers the usual complex arithmetic): - Sum: (((a,-b),(b,a))+((c,-d),(d,c))=((a+c,-(b+d)),(b+d,a+c))) - Product: (((a,-b),(b,a))((c,-d),(d,c))=((ac-bd, -(ad+bc)),(ad+bc, ac-bd))) - Units: (1=((1,0),(0,1))), (i:=((0,-1),(1,0))), (i^2=-1)

## 8. Containment of Number Systems

[  (repeated application) ;;  (signed naturals) ;;  (fractions of integers) ;; M (rational near‑limits: upper/lower layers) ;;  (fractions of (M)) ;;  (matrix form of ()) ]

Note. (M,,) are structures specific to this work; extensions to new quaternions or octonions are possible.

## Column: On Decimal Notation

Finite decimals and repeating decimals (e.g., 0.333…) are human‑readable with …, but **identity as numbers** depends on the ground field. Here we **do not adopt** the classical identity (0.999=1); we treat it as the **left neighbor class** 1♭♭ in our setting.

## Appendix A: Notes for a λ‑Calculus Interpreter

* Data types in this chapter:
  + Nat := repeat f (0=f(), succ(n)=f(n))
  + Int := (sgn∈{0,1}, Nat)
  + Rat := (Int, Int≠0) with normalization
  + Rel := Z (layer index for …♭♭, a, a##, …)
  + M := (Rel, Rat) where Rel is an integer layer; order is lexicographic
  + CR := Frac(M) (integer scaling normalization + reduction)
* Formatting: line breaks allowed inside formulas, ; at end of a displayed formula; comments start after ;.
* Example (upper representatives for a=5):
  + ((1·5+1)/1) = 6/1; ((2·5+1)/2) = 11/2; … is a monotone decreasing family approaching 5.

## Appendix B: Samples

**Inequalities for neighbors.** (a^{(-2)} < a^{(-1)} < a < a^{(+1)} < a^{(+2)}).

**“Add/remove symbol” identities.** (1{(-1)}+1{(+1)}=2), (0{(+2)}-0{(+3)}=0^{(-1)}).

**Non‑reducible examples.** (1^{(+1)}/2), (1^{(+1)}/3), etc.

*(end)*